

MODEL OF DISPERSION TAKING INTO ACCOUNT THE DIFFERENCE IN VISCOSITIES OF THE MEDIA

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A one-dimensional model of admixture dispersion in a turbulent stream in a tube is proposed, taking into account the difference in the viscosities of the fluids being mixed.

Let us consider the combined flow of two mutually soluble fluids during which the formation of a mixing region separating the homogeneous stream components occurs. The distribution of the mean concentration over the tube cross section in the mixing region can be described by a one-dimensional equation of the heat conduction type with some effective coefficient

$$\frac{\partial \theta}{\partial \tau} = \frac{\partial}{\partial \zeta} \left(K_* \frac{\partial \theta}{\partial \zeta} \right), \quad \theta = \theta(\tau, \zeta), \quad (1)$$
$$K_* = K_* \left(\frac{\rho U d}{\mu_1}, \frac{\mu_1}{\mu_2}, \theta, \frac{\partial \theta}{\partial \zeta}, \dots \right).$$

As tests show, this coefficient differs from the molecular and turbulent transport coefficients. Its dimensionless value depends on a number of parameters such as the Reynolds number calculated for one of the fluids, the roughness, the ratio between the fluid densities and viscosities, as well as the concentration, concentration gradients, etc. These dependences can be established by using either tests or on the basis of a theoretical solution of the three-dimensional mixing problem in a stream in a tube.

From this viewpoint, a one-dimensional model of dispersion with an effective coefficient dependent only on the Reynolds number is constructed theoretically in the known paper of Taylor [1]. This latter circumstance is due to the fact that Taylor took the density and viscosity of the fluids being mixed as identical in constructing the model.

If these assumptions are rejected, then the one-dimensional model of dispersion is more complex since, in this case, the effective coefficient will depend on the concentration distribution in the mixture region. Let us estimate the influence of the difference between the fluid viscosities on the quantity of mixture being formed and on the concentration distribution. To this end, let us consider the fluids being mixed in the stream in the tube to have identical densities but distinct viscosities. Such an assumption corresponds well enough to the case of the motion of two diverse petroleum products whose densities differ to a considerably lesser extent than do the viscosities.

Let us consider the concentration distribution for times much greater than the diffusion d^2/D_0 , when the length of the mixing region becomes much greater than the tube diameter. Because of the radial diffusion substantial inhomogeneities in the concentration in the tube cross section vanish for such times and only insignificant deviations of the values of the concentration from the mean values in the tube cross section will exist in the mixing region (these deviations are due to the inhomogeneous convective transfer because of the velocity profile).

For an observer moving at the mean stream velocity in such a mixing region, the concentration profile hardly varies with time. Hence, the local time derivative of the concentration can be discarded in the equation describing the concentration distribution in a moving coordinate system. We have

$$(u - U) \frac{\partial \theta}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left[r \epsilon(r) \frac{\partial \theta}{\partial r} \right]. \quad (2)$$

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The derivative c'_x is replaced by θ'_x in the left side of this equation because of the smallness of the radial inhomogeneities. The solution of (2) which satisfies the condition that streams of material on the inner tube surface are zero is

$$c = c_0 - \frac{\partial \theta}{\partial x} \int_r^a \frac{1}{re(r)} \left(\int_0^r (u-U)\eta d\eta \right) dr. \quad (3)$$

The function $c_0(t, x)$ can be determined in terms of the distribution of the mean stream concentration. We have

$$c_0 = \theta + \frac{1}{a^2} \frac{\partial \theta}{\partial x} \int_0^a \frac{r dr}{e(r)} \left(\int_0^r (u-U)\eta d\eta \right). \quad (4)$$

The effective coefficient of turbulent diffusion [2] can be found by using the distribution (3). We have

$$K = 2 \int_0^a \frac{dr}{re(r)} \left[\int_0^r (u-U)\eta d\eta \right]^2. \quad (5)$$

Two unknown functions, the average stream velocity and the coefficient of turbulent diffusion, are in this formula. Material and momentum transfer occur with almost equal intensity in a turbulent flow, hence, we assume the transfer coefficients D_t and ν_t identical. Then the quantity D_t in a stream in a tube can be calculated by means of the following

$$D_t = -u_*^2 \frac{r}{a} \left(\frac{\partial u}{\partial r} \right)^{-1}. \quad (6)$$

To determine the velocity profile, let us take a two-layer scheme for a turbulent stream in a smooth tube, a laminar layer near the inner tube surface and a turbulent layer in which the stream molecular viscosity plays no substantial role in the formation of the velocity profile. The whole influence of the molecular viscosity on the velocity distribution and other stream parameters is localized in the laminar layer in this scheme.

The equation to determine the velocity in a laminar layer is

$$\frac{\partial u}{\partial y} = \frac{u_*^2}{\nu}, \quad v = v(c, \nu_1, \nu_2), \quad 0 < y < y_+. \quad (7)$$

The equation differs from the analogous equation for a homogeneous stream in that the viscosity of an inhomogeneous fluid is a variable in it, dependent on the concentration distribution. Hence, it is impossible to solve (7) without knowing the concentration distribution which, in turn, depends on the velocity of the inhomogeneous stream.

Let us expand the function of the concentration $c(t, x, y)$ in a Taylor series in y in the neighborhood of the point $y = 0$:

$$c(t, x, y) = c_0(t, x) + \left(\frac{\partial^2 c}{\partial y^2} \right)_0 \frac{y^2}{2!} + \dots \quad (8)$$

By using (3), let us estimate the magnitude of the second term in this expansion within the limits of the laminar layer

$$\left| \left(\frac{\partial^2 c}{\partial y^2} \right)_0 \frac{y_+^2}{2!} \right| \leq \frac{1}{2} \frac{\tau_d}{\tau_h} \left| \frac{\partial \theta}{\partial \xi} \right|. \quad (9)$$

It follows from this estimate that the value of the second member depends on the relationship between the characteristic time intervals associated with the convective transfer and the molecular diffusion in the laminar layer.

Let us henceforth consider the time interval within which the concentration changes essential because of convective transfer to be an order of magnitude greater than the diffusion constant ($\tau_K/\tau_\alpha \gg 1$). In this case the concentration in the laminar layer can be taken approximately equal to $c_0(t, x)$.

Taking this approximation, we determine the velocity profile in the laminar layer

$$u = \frac{u_*^2}{\nu} y, \quad v = v(c_0, \nu_1, \nu_2), \quad 0 \leq y \leq y_+. \quad (10)$$

To determine the velocity profile in the core of the turbulent stream, let us make assumptions which result in a logarithmic velocity profile. Let us assume the friction stress in the turbulent layer to be constant and equal to the friction stress on the inner tube surface. Let us define the turbulent stream viscosity by the Prandtl formula by considering $t = \kappa y$:

$$v_t = l^2 \frac{\partial u}{\partial y}, \quad y_+ < y < a, \quad \kappa = 0.4. \quad (11)$$

Under the assumptions made, the dimensionless velocity profile is

$$\varphi = 2.5\alpha \ln \frac{\eta}{\eta_+} + \varphi_+, \quad \eta > \eta_+. \quad (12)$$

This profile differs from the analogous velocity profile for a homogeneous stream in a tube by the fact that the parameters α , η_+ and φ_+ are functions of the variables τ and ξ since they depend on the mean concentration distributions. The expansion for the fluid velocity profile in the mixing region is used to calculate the quantities K_* and c_0 in the turbulent stream core. We have

$$K_* = 2.82\alpha, \quad c_0 = \theta + 3.9 \frac{\partial \theta}{\partial \xi} \approx \theta. \quad (13)$$

In contrast to the analogous Taylor formula for K_* , the quantity α and therefore, the effective coefficient of diffusion K_* evaluated by means of (13), depend on the concentration distribution.

Using (12), let us evaluate the mean stream velocity and obtain an equation to determine α as a function of $\theta(\tau, \xi)$:

$$1 = \beta\alpha + 2.5\alpha \left(\ln \frac{\alpha \text{Re}}{2\beta} - \frac{3}{2} \right). \quad (14)$$

The Reynolds number in this formula is evaluated in terms of the viscosity $\nu(\theta)$

$$\text{Re} = (v_1/\nu(\theta, v_1, v_2)) \text{Re}_1. \quad (15)$$

We evaluate the kinematic viscosity $\nu(\theta)$ by means of the formula [3]

$$\frac{1}{\nu^3} = \frac{1}{\nu_1^3} \theta + \frac{1}{\nu_2^3} (1 - \theta). \quad (16)$$

Taking these last two equations into account, (14) can be rewritten as

$$1 = \beta\alpha + 2.5\alpha \left(\ln \frac{\alpha \text{Re}_1}{2\beta} - 3 \ln \left(\theta + \nu_0^{\frac{1}{3}} (1 - \theta) \right) - \frac{3}{2} \right). \quad (17)$$

Now (14) for a fixed value of θ , agrees with the theoretical Prandtl formula to calculate the hydraulic drag of a homogeneous stream whose viscosity equals the viscosity of the mixture for the selected value of θ . Within a broad range of Reynolds numbers, the dependence of α on Re , determined by the Prandtl formula, can be approximated by the Colebrook function

$$\alpha = \left(3.6 \sqrt{2} \lg \frac{1}{7} \text{Re} \right)^{-1}. \quad (18)$$

If it is assumed that $\text{Re} = (\nu_1/\nu) \text{Re}_1$, this same dependence will approximate the relation between α and Re determination by (14) for each fixed value of θ .

In contrast to (14), the dependence (18) is explicit and its use will simplify the dispersion model substantially.

Using (18), let us write the explicit concentration dependence of the effective coefficient of diffusion

$$K_* = \Gamma \left[1 - \frac{3}{\ln \frac{1}{7} \text{Re}_1} \ln \left(\theta + \nu_0^{\frac{1}{3}} (1 - \theta) \right) \right]^{-1}, \quad (19)$$

$$\Gamma = 1.28 \left(\ln \frac{1}{7} \text{Re}_1 \right)^{-1}, \quad \nu_0 = \frac{\nu_2}{\nu_1}.$$

In order to compare the concentration distributions in the mixing regions evaluated for $\nu_0 < 1$ and $\nu_0 > 1$, let us find the solution to (1) for the problem of successive repumping of two fluids, when one moves after the other.

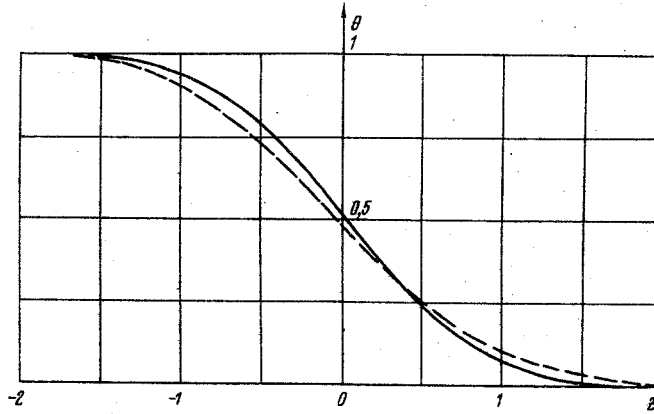


Fig. 1. Concentration distribution $\theta(z)$ for the fluids following in a different order. (The solid curve corresponds to the parameters $\nu_2/\nu_1 = 0.435$, $Re_1 = 6 \cdot 10^4$, and the dashed curve to $\nu_2/\nu_1 = 23$, $Re_1 = 1.3 \cdot 10^6$).

The limit conditions for this problem are

$$\theta(0, \zeta) = 0, \theta(\tau, -\infty) = 1, \theta(\tau, +\infty) = 0. \quad (20)$$

The problem admits of a self-similar solution dependent on one variable $z = \xi/2\sqrt{\Gamma\tau}$. The desired solution satisfies the following equation and boundary conditions:

$$\begin{aligned} -2z \frac{d\theta}{dz} &= \frac{d}{dz} \left[\Phi(\theta) \frac{d\theta}{dz} \right], \theta(-\infty) = 1, \theta(+\infty) = 0, \\ \Phi &= (1 - b \ln(\theta + \nu_0^{1/3}(1 - \theta)))^{-1}, \\ b &= 3 \left(\ln \frac{1}{7} Re_1 \right)^{-1}. \end{aligned} \quad (21)$$

To solve the problem, let us use the method described in [5]. Let us introduce the new function $f = 1 - \theta$ and the independent variable $z_* = z + z_0$. Let us rewrite (21) thus

$$\begin{aligned} \frac{d}{dz_*} \left[(1 - b \ln(1 - \gamma f))^{-1} \frac{df}{dz_*} \right] + 2(z_* - z_0) \frac{df}{dz_*} &= 0, \\ \gamma &= 1 - \nu_0^{1/3}, f = f(z_*). \end{aligned} \quad (22)$$

Let us transfer the limit condition for $z = -\infty$ to the point $z = -z_0$. Then the limit conditions for the function $f(z_*)$ become

$$f(0) = 0, f(+\infty) = 1. \quad (23)$$

Let us introduce the linear transformation

$$f = A^{\alpha_1} h, z_* = A^{\alpha_2} z_1$$

and let us give still another boundary condition

$$\frac{df(0)}{dz_*} = A. \quad (24)$$

Let us take the exponents α_1 and α_2 in such a way that (22) for the function $h(z_1)$ and condition (24) would be independent of A . To do this we set $\alpha_1 = 1$, $\alpha_2 = 0$ and introduce a new constant $\delta = \gamma A$. Then we solve the Cauchy problem with the following limit conditions

$$h(0) = 0, \frac{dh(0)}{dz_1} = 1. \quad (25)$$

for (22) with the unknown function $h(z_1)$ for some fixed value of the parameter δ . For the solution found from the condition at $z_1 = \infty$ we determine the constant $A = (h(\infty))^{-1}$ and we calculate the desired function $\theta = 1 - (h(\infty))^{-1} h(z_1 - z_0)$. The quantity A depends on the parameter δ , which we variate in such a way as to

find the solution of the initial problem for a given value of ν_0 . Computations have shown that the quantity z_0 should be taken equal to 2.5. For $z_0 > 2.5$ the solutions $\theta(z)$ are practically identical. Represented graphically in the figure are solutions of the problem (21) for the following set of parameters: $\nu_0 = 23$, $Re_1 = 1.3 \cdot 10^6$ and $\nu_0 = 0.435$, $Re_1 = 6 \cdot 10^4$. The change in the concentration distribution as a function of the order in which the fluids follow can be traced for identical pipeline productivity by means of these graphical solutions. (The subscript 1 refers to the parameters of the fluid moving from behind.)

The quantity of one fluid incident on the other can be estimated by means of the magnitudes of the area over the curve $\theta(z)$ in the region $z < 0$ and under this curve for $z > 0$. The sum of these areas characterizes the total quantity of fluid being mixed. In the case under consideration this sum is 10% greater if the more viscous fluid moves in front. It hence follows that the quantity of mixture being formed depends on the order in which the fluids follow and it will be greater when the more viscous fluid moves in front.

A model of mixing fluids with different viscosities was considered above in which it is assumed that the equalization of the concentrations occurs instantaneously ($\tau_k \gg \tau_d$). Another limit case would be of interest, when $\tau_k \ll \tau_d$ and the concentration of the fluid moving in front in the laminar layer does not succeed in being changed during the passage of the mixture through the section selected. In this case, we should set $\theta = 0$ in (19) for the calculation of the effective diffusion coefficient. A linear equation is hence obtained for the concentration distribution. If the quantity of admixture is compared as a function the order of succession within the framework of this model then it turns out that the total quantity of the admixture in each of the fluids is 15% greater for the same values of the parameters ν_0 and Re_1 if the less viscous fluid moves from behind than for a reverse order of succession when the less viscous fluid moves in front.

Therefore, on the basis of the one-dimensional diffusion model taking account of the difference in the fluid viscosities, we have succeeded in showing in what way the order of fluid succession influences the quantity of mixture being formed and the concentration distribution. Qualitatively similar results have been obtained in [4] in which a model taking account of the difference in viscosities has been constructed on the basis of tests.

NOTATION

θ	is the mean concentration over the tube cross section;
$\tau = Ut/a$;	
t	is the time;
U	is the mean stream velocity;
$\xi = x/a$	is the distance in a moving coordinate system;
a	is a tube radius;
$K_* = K/Ud$;	
K	is the effective coefficient of diffusion;
μ_1 and ν_1	are the fluid viscosities;
D_0	is the typical value of the coefficient of turbulent diffusion;
$u(r)$	is the average stream velocity;
r	is the distance along the radius;
$\varepsilon = D + D_t$;	
D and D_t	are the coefficients of molecular and turbulent diffusion;
c	is the average concentration;
c_0	is the value of the concentration on the inner tube surface;
u_*	is the dynamic turbulent velocity;
y	is the distance from the inner tube surface;
y_+	is the laminar layer thickness;
ν	is the mixture viscosity;
ν_t	is the turbulent viscosity;
$\xi = x/L$;	
L	is the length of mixture region;
$\varphi = u/U$;	
$\alpha = u_*/U$;	
$\eta = y/a$;	
φ_+	is the dimensionless velocity on the laminar layer boundary;
$Re_1 = Ud/\nu_1$;	
$\nu_0 = \nu_2/\nu_1$;	
$z = \xi/2\sqrt{1\tau}$;	

$$\begin{aligned}\tau_k &= L/U; \\ \tau_d &= y_+^2/D; \\ y_+ &= 11.5\nu/u_*; \\ \beta &= 11.5.\end{aligned}$$

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